Calderón-Zygmund Theory of Singular Integrals Handout

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1 Singular Integrals

Definition 1.1. Let $K : \mathbb{R}^d \backslash \{0\} \to \mathbb{C}$ satisfy, for some constant B,

1. $$|K(x)| \leqslant B|x|^{-d},$$
2. $$\int_{\{|x|>2|y|\}} |K(x) - K(x-y)|dx \leqslant B, \text{ for all } y \neq 0,$$
3. $$\int_{r<|x|<s} K(x)dx = 0, \text{ for all } 0 < r < s < \infty.$$

Then, K is called a **Calderón-Zygmund Kernel**.

With such a kernel K one can associate a translation invariant operator:

$$Tf(x) := \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} K(x-y)f(y)dy,$$

for $f \in \mathcal{S}(\mathbb{R}^d)$. An operator of this type is called a **singular integral operator**.

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2 Main Theorem

Theorem 2.1. *For every $1 < p < \infty$ one can extend any singular integral operator T to an operator bounded from $L^p \to L^p$.*

The steps of the proof are as follows:

- Show that T is bounded from $L^2 \to L^2$ using the properties of the kernel and Plancherel's theorem.
- Show that T is bounded from $L^1 \to L^{1,\infty}$ via a Calderón-Zygmund decomposition.
- Interpolate between these to bounds to conclude that T is bounded from $L^p \to L^p$ for $1 < p \leqslant 2$.
- Apply the previous results to the adjoint operator T^* (with $K^*(x) = \bar{K}(-x)$) to conclude that T is bounded from $L^p \to L^p$ for $1 < p < \infty$. (i.e. $\langle Tf, g\rangle = \langle f, T^*g\rangle$ use duality).

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3 Calderón-Zygmund Decomposition

3.1 Dyadic Intervals

Definition 3.1. A **dyadic** interval in $\mathbb{R}$ is an interval of the form

$$[2^k m, 2^k(m+1)) \text{ for } k, l \in \mathbb{Z}.$$

If two dyadic intervals intersect, then they are equal OR one is contained inside the other. Dyadic boxes in $\mathbb{R}^d$ are products of the intervals given in the above definition.

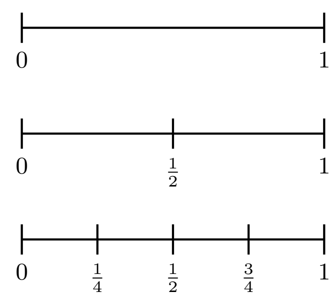

Figure 3.1.1: dyadic intervals at various scales

3.2 Summary of Decomposition

Fix $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$. We will write f as a sum of a "good" function and a "bad" function. The good function will be bounded by λ. The bad function can be large, but it will be supported in a set of controlled measure and it will have mean zero on its support.

- Select dyadic intervals J_i such that

$$\frac{1}{|J_i|}\int_{J_i} |f(x)|dx > \lambda$$

  and such that J_i is **maximal** with respect to inclusion. Denote by Ω the union of all such J_i.

- Write $f = g + b$ where

$$g = f\chi_{\Omega^c} + \sum_{J_i}\left(\frac{1}{|J_i|}\int_{J_i} f(x)dx\right)\chi_{J_i}$$

  and

$$b = f - g = \sum_{J_i}\left(f - \frac{1}{|J_i|}\int_{J_i} f(x)dx\right)\chi_{J_i} := \sum_{J_i} b_{J_i}.$$

- With this decomposition of f we get the following estimates:

  –
$$||g||_\infty \lesssim \lambda$$

  –
$$||g||_1 \leqslant ||f||_1$$

  –
$$||b||_1 \leqslant 2||f||_1$$

  –
$$\int_{J_i} b_{J_i}(x)dx = 0$$

  –
$$|\Omega| \leqslant \frac{1}{\lambda}||f||_1.$$
